MR4773115 05C40 05C38 05C45 05C75 05D05

Leader, Imre (4-CAMB); **Ranelović, Žarko** (4-CAMB);
Tan, Ta Sheng (MAL-MALAS-IM)

A note on Hamiltonian-intersecting families of graphs. (English. English summary)

Discrete Math. **347** (2024), *no.* 11, *Paper No.* 114160, 4 *pp.*

All graphs considered in this paper are simple. Let $\mathcal{F}$ be a family of undirected graphs on vertex set $[n] := \{1, \dots, n\}$. Say that $\mathcal{F}$ is $\mathcal{P}$*-intersecting* for a graph property $\mathcal{P}$ if $F \cap F'$ satisfies the property $\mathcal{P}$ for all distinct $F, F' \in \mathcal{F}$. The natural extremal problem is to find the maximum size of a $\mathcal{P}$-intersecting family over $[n]$. Classically, many extremal problems of this kind have been studied where $\mathcal{P}$ is the property "contains a copy of Γ" for some fixed graph Γ. For instance, an important conjecture of Simonovits–Sós from 1976 stated that the maximum size of a triangle-intersecting family of graphs over $[n]$ is $2^{\binom{n}{2}-3}$; this was proved by D. C. Ellis, Y. Filmus and E. Friedgut [J. Eur. Math. Soc. (JEMS) **14** (2012), no. 3, 841–885; MR2911886] by spectral and Fourier analytic methods. In this short paper, the authors consider the following problems, motivated by the work of A. Berger et al. ["Connected-intersecting families of graphs", preprint, arXiv:1901.01616], who showed that any connected-intersecting family $\mathcal{F}$ over $[n]$ satisfies $|\mathcal{F}| \le 2^{\binom{n}{2}-(n-1)}$.

First, let $\mathcal{F}$ be a family of directed graphs over $[n]$ that is strongly connected-intersecting. Then, the authors show that $|\mathcal{F}| \le 2^{n(n-1)-n}$. This bound is seen to be tight by considering the family of all directed graphs containing a fixed directed n-cycle.

Next, let $\mathcal{F}$ be a family of oriented graphs over $[n]$ that is strongly connected-intersecting, where an oriented graph is a directed graph obtained by orienting the edges of an undirected graph. Then, the authors show that $|\mathcal{F}| \le 3^{\binom{n}{2}-n}$. Here, equality occurs if and *only if* $\mathcal{F}$ consists of all oriented graphs containing a fixed directed n-cycle.

Next, let $\mathcal{F}$ be a family of undirected graphs over $[n]$ that is Hamiltonian-intersecting. Then, $|\mathcal{F}| \le 2^{\binom{n}{2}-n}$, and as before, equality occurs if and only if $\mathcal{F}$ consists of all undirected graphs containing a fixed n-cycle. Essentially the same proof shows that the same bound holds for any family $\mathcal{F}$ of undirected graphs over $[n]$ that is biconnected-intersecting.

The authors then raise the following conjectures for families of undirected graphs over $[n]$. It is conjectured that if $\mathcal{F}$ is a family of graphs that is 2-edge-connected-intersecting, then $|\mathcal{F}| \le 2^{\binom{n}{2}-n}$. This is based on the hypothesis that the best case scenario is when $\mathcal{F}$ consists of all graphs containing a fixed n-cycle.

Next, it is conjectured that if $\mathcal{F}$ is a family of graphs such that the intersection of any two members of $\mathcal{F}$ has at most two connected components, then $|\mathcal{F}| \le (n+1)2^{\binom{n}{2}-n}$. This is based on the hypothesis that (one of) the best case scenario(s) is when $\mathcal{F}$ consists of all graphs containing all but (at most) one of the edges of a fixed n-cycle.

Lastly, it is hypothesized that if "two connected components" is replaced by "k connected components" in the previous conjecture for $k \ge 3$, then the best case scenario is (for large enough n) to fix a graph H containing $k-1$ cycles of balanced lengths meeting at a single vertex, and then let $\mathcal{F}$ consist of all graphs containing all but (at most) $k-1$ edges of H, one from each cycle. Specifically, when $k=3$ and n is odd, it is conjectured that $|\mathcal{F}| \le \frac{(n+3)^2}{8} 2^{\binom{n}{2}-n}$.

The proofs are short and interesting applications of linear algebraic methods, and the uniform covers theorem of B. Bollobás and A. Thomason [Bull. London Math. Soc. **27** (1995), no. 5, 417–424; MR1338683]. These results add to the vast body of literature on Erdős–Ko–Rado-type theorems for intersecting graph families.

The proof of the case when $\mathcal{F}$ is a family of oriented graphs that is strongly connected-intersecting has a couple of typos. Here, the authors find that $|\pi(\mathcal{F})| = 3^{(n-1)\alpha}$ for $\alpha = 1 - \frac{2}{n-1}$, where π is the projection onto the set of $2(n-1)$ directed edges on $[n]$ that are incident on a fixed vertex i. Then, they apply the uniform covers theorem of Bollobás and Thomason to conclude that $|\mathcal{F}| = 3^{\binom{n}{2}\alpha}$. However, in order to apply the uniform covers theorem, one needs to express $|\pi(\mathcal{F})|$ as $3^{2(n-1)\beta}$ for $\beta = \frac{1}{2} - \frac{1}{n-1}$, and then conclude that $|\mathcal{F}| = 3^{n(n-1)\beta}$.

Furthermore, while it is true that any extremal family $\widetilde{\mathcal{F}} \subset (0,1,2)^{\binom{n}{2}}$ is a product of n singletons and $\binom{n}{2} - n$ sets of size 3, it is not completely clear that this deserves to be called a "cuboid, that is, a product of 1-dimensional sets". For, the equivalence classes of the uniform cover here are the $\binom{n}{2}$ two-element sets $\{(i,j),(j,i)\}$, so the corresponding body $K_{\mathcal{F}} \subset \mathbb{R}^{n(n-1)}$ is a product of the projections of $K_{\mathcal{F}}$ onto the two-dimensional planes corresponding to these coordinates. In particular, there are $\binom{n}{2} - n$ such planes where the projection of $K_{\mathcal{F}}$ has area 3, but is not a rectangle; so, $K_{\mathcal{F}}$ is not a cuboid in $\mathbb{R}^{n(n-1)}$ when $\mathcal{F}$ is extremal. *Brahadeesh Sankarnarayanan*

[References]

1. A. Berger, R. Berkowitz, P. Devlin, M. Doppelt, S. Durham, T. Murthy, H. Vemuri, Connected-intersecting families of graphs, arXiv:1901.01616.
2. B. Bollobás, A. Thomason, Projections of bodies and hereditary properties of hypergraphs, Bull. Lond. Math. Soc. 27 (5) (1995) 417–424. MR1338683
3. F. Chung, R. Graham, P. Frankl, J. Shearer, Some intersection theorems for ordered sets and graphs, J. Comb. Theory, Ser. A 43 (1) (1986) 23–37. MR0859293
4. D. Ellis, Intersection problems in extremal combinatorics: theorems, techniques and questions old and new, in: Surveys in Combinatorics 2022, Lond. Math. Soc. Lect. Note Ser. 481 (2022) 115–173. MR4421401
5. P. Frankl, N. Tokushige, The Erdős–Ko–Rado Theorem for integer sequences, Combinatorica 19 (1) (1999) 55–63. MR1722210

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.